Political Power and Socio-Economic Inequality

An Application of the Canonical Ensemble to Social Sciences

Daniel Kraft

July 25th, 2012

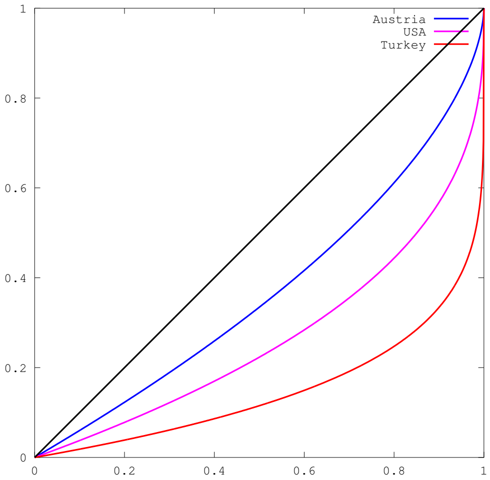
Overview

The Model

Social Inequality

"In 2010, average real income per family [in the United States] grew by 2.3 % but the gains were very uneven. Top 1 % incomes grew by 11.6 % while bottom 99 % incomes grew only by 0.2 %. Hence, the top 1 % captured 93 % of the income gains in the first year of recovery."

Social Inequality



The Social Space

Individuals described by *three dimensions*:

The Social Space

Individuals described by *three dimensions*:

Labour $a \in [0, 1]$

Income $I \in [L, \infty)$

to model the economy.

The Social Space

Individuals described by *three dimensions*:

Power $m \in [0, 1]$
to model possibly unfair political decisions, and

Labour $a \in [0, 1]$

Income $I \in [L, \infty)$

to model the economy.

The Social Space

Individuals described by *three dimensions*:

Power $m \in [0, 1]$
to model possibly unfair political decisions, and

Labour $a \in [0, 1]$
Income $I \in [L, \infty)$
to model the economy.

Definition

My social space: $U = [0, 1] \times [0, 1] \times [L, \infty)$
Individuals: $x = (m, p) = (m, a, I) \in U$

Strain Functions

Individuals try to **maximise their personal happiness**, respectively minimise their *strain*:

Strain Functions

Individuals try to **maximise their personal happiness**, respectively minimise their *strain*:

Definition

$f : [0,1] \times [L, \infty) \to \mathbb{R} \cup \{\infty\}$ is a strictly convex *strain function*:

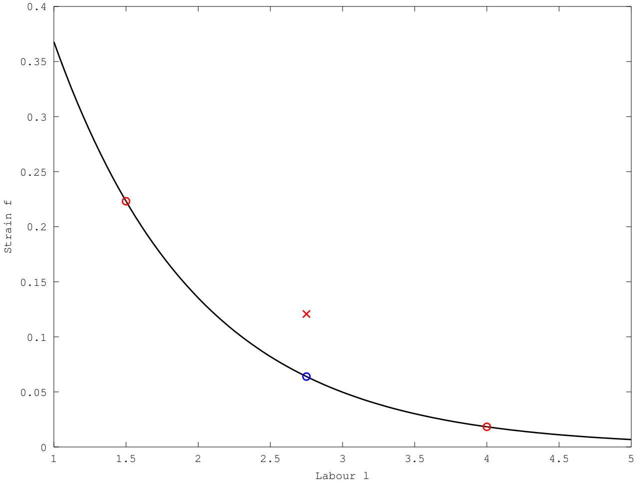
- f possesses certain regularity,
- f is strictly increasing in a and decreasing in l, and
- f is strictly convex.

Strain Functions

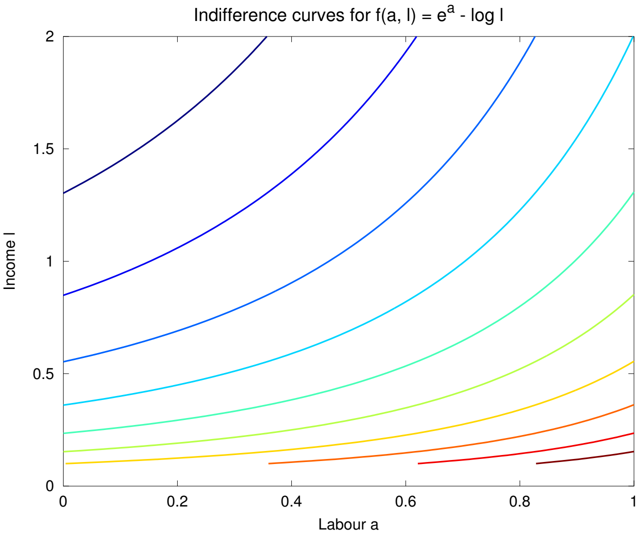
A note on convexity, a. k. a. *decreasing marginal utility*:

Strain Functions

A note on convexity, a. k. a. *decreasing marginal utility*:



Strain Functions



Coupling the Individuals

Of course, single individuals do not yet form a society!

Coupling the Individuals

Of course, single individuals do not yet form a society!

We require a *closed economy*: $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} l_n$

Normalisation of powers: $\sum_{n=1}^{N} m_n = 1$

Coupling the Individuals

Of course, single individuals do not yet form a society!

We require a *closed economy*: $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} l_n$

Normalisation of powers: $\sum_{n=1}^{N} m_n = 1$

Definition

$\Omega \subset U^N$ is the set of all configurations $X = (x_1, \ldots, x_N)$ that satisfy these conditions. x_i are the individuals in my social space.

"Dynamics" of the System

Definition

We define the abstract *energy* $\mathcal{H} : \Omega \to \mathbb{R} \cup \{\infty\}$:

$$\mathcal{H}(X) = \sum_{n=1}^{N} \left(\frac{\gamma}{N} + (1-\gamma) m_n \right) f(a_n, l_n),$$

where $\gamma \in [0, 1]$.

"Dynamics" of the System

Definition

We define the abstract *energy* $\mathcal{H} : \Omega \to \mathbb{R} \cup \{\infty\}$:

$$\mathcal{H}(X) = \sum_{n=1}^{N} \left(\frac{\gamma}{N} + (1-\gamma) m_n \right) f(a_n, l_n),$$

where $\gamma \in [0, 1]$.

Assume that the system tries to **minimise** $\mathcal{H}$ **over** Ω.

"Dynamics" of the System

For a temperature $T > 0$ (or equivalently $\beta = \frac{1}{kT} > 0$) assume a **Boltzmann distribution** (canonical ensemble):

"Dynamics" of the System

For a temperature $T > 0$ (or equivalently $\beta = \frac{1}{kT} > 0$) assume a **Boltzmann distribution** (canonical ensemble):

Definition

For $A \subset \Omega$, define its probability as

$$\pi_T(A) = \frac{1}{\mathcal{Z}} \int_A e^{-\beta \mathcal{H}(X)} \, dX,$$

where

$$\mathcal{Z} = \int_\Omega e^{-\beta \mathcal{H}(X)} \, dX.$$

Theoretical Analysis

Structure of the Minimum

Theorem

Let $\gamma = 1$. Then $X \in \Omega$ is a global minimum of $\mathcal{H}$ over Ω iff

$$a_n = l_n = a^*, \text{ for all } n = 1, \ldots, N.$$

a^* is the minimum of $a \mapsto f(a, a)$ over $[L, 1]$.

Structure of the Minimum

Theorem

Let $\gamma = 1$. Then $X \in \Omega$ is a global minimum of $\mathcal{H}$ over Ω iff

$$a_n = l_n = a^*, \text{ for all } n = 1, \dots, N.$$

a^* is the minimum of $a \mapsto f(a, a)$ over $[L, 1]$.

Theorem

Let $\gamma < 1$, then $X^* \in \Omega$ of the form

$$X^* = ((1, a_1^*, l_1^*), (0, a_0^*, l_0^*), \dots, (0, a_0^*, l_0^*))$$

minimises $\mathcal{H}$ over Ω. $a_0^*, a_1^* \in [0, 1]$ and $l_0^*, l_1^* \geq L$ depend on f and the parameters.
This minimum is unique up to permutation of the individuals.

A Simplified Problem

$$\min_{a_0, l_0, a_1, l_1} \gamma \frac{N-1}{N} f(a_0, l_0) + \left(\frac{\gamma}{N} + (1-\gamma) \right) f(a_1, l_1),$$

where $a_0, a_1 \in [0,1]$, $l_0, l_1 \geq L$ and

$$(N-1)a_0 + a_1 = (N-1)l_0 + l_1.$$

A Simplified Problem

$$\min_{a_0, l_0, a_1, l_1} \gamma \frac{N-1}{N} f(a_0, l_0) + \left(\frac{\gamma}{N} + (1-\gamma) \right) f(a_1, l_1),$$

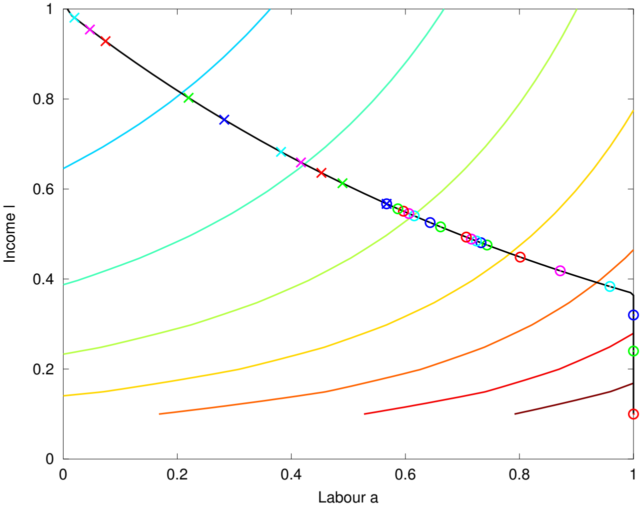
where $a_0, a_1 \in [0,1]$, $l_0, l_1 \geq L$ and

$$(N-1)a_0 + a_1 = (N-1)l_0 + l_1.$$

Can be solved for instance by:

- Gradient projection techniques, or
- Newton's method applied to the Lagrangian.

A Simplified Problem



Further Results

Consider the simplified problem.

Theorem

$$f(a_1, l_1) \leq f(a_0, l_0)$$

If $\gamma < \gamma'$, *we also have* $f(a_0, l_0) \geq f(a'_0, l'_0)$ *and* $f(a_1, l_1) \leq f(a'_1, l'_1)$.

Further Results

Consider the simplified problem.

Theorem

$$f(a_1, l_1) \leq f(a_0, l_0)$$

If $\gamma < \gamma'$*, we also have* $f(a_0, l_0) \geq f(a_0', l_0')$ *and* $f(a_1, l_1) \leq f(a_1', l_1')$*.*

Let f be everywhere finite.

Theorem

The minimiser $(a_0, l_0, a_1, l_1) \in \mathbb{R}^4$ *depends continuously on* γ*.*

Further Results

Consider the simplified problem.

Theorem

$$f(a_1, l_1) \leq f(a_0, l_0)$$

If $\gamma < \gamma'$*, we also have* $f(a_0, l_0) \geq f(a_0', l_0')$ *and* $f(a_1, l_1) \leq f(a_1', l_1')$.

Let f be everywhere finite.

Theorem

The minimiser $(a_0, l_0, a_1, l_1) \in \mathbb{R}^4$ *depends continuously on* γ.

Theorem

If $\gamma < 1$*, we have* $a_0 > l_0$ *and* $a_1 < l_1$.

Simulation Results

Metropolis Algorithm

Calculation of $\mathcal{Z}$ and expectation values intractable!
$\rightarrow$ Numerical simulation, Monte-Carlo method

Metropolis Algorithm

Calculation of $\mathcal{Z}$ and expectation values intractable!
$\rightarrow$ Numerical simulation, Monte-Carlo method

Custom Metropolis algorithm:

- Generate configurations sampled by π_T.

Metropolis Algorithm

Calculation of $\mathcal{Z}$ and expectation values intractable!
$\rightarrow$ Numerical simulation, Monte-Carlo method

Custom Metropolis algorithm:

- Generate configurations sampled by π_T.
- Markov process, updating "current" configuration.
- We need $\frac{P(X')}{P(X)}$, but not $P(X)$ directly.
  $\rightarrow$ $\mathcal{Z}$ drops out!

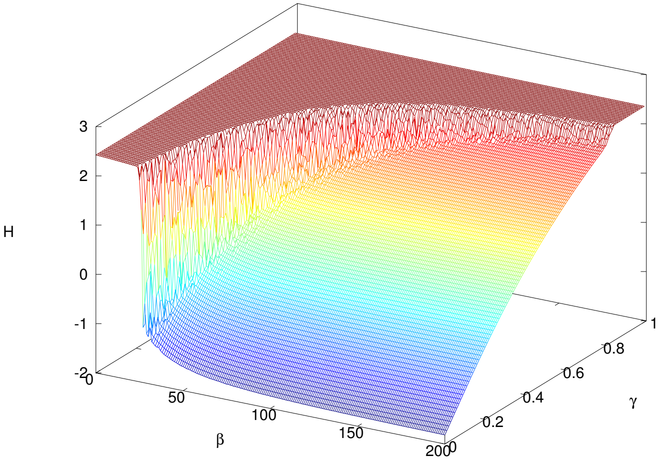
Metropolis Algorithm

Calculation of $\mathcal{Z}$ and expectation values intractable!
$\rightarrow$ Numerical simulation, Monte-Carlo method

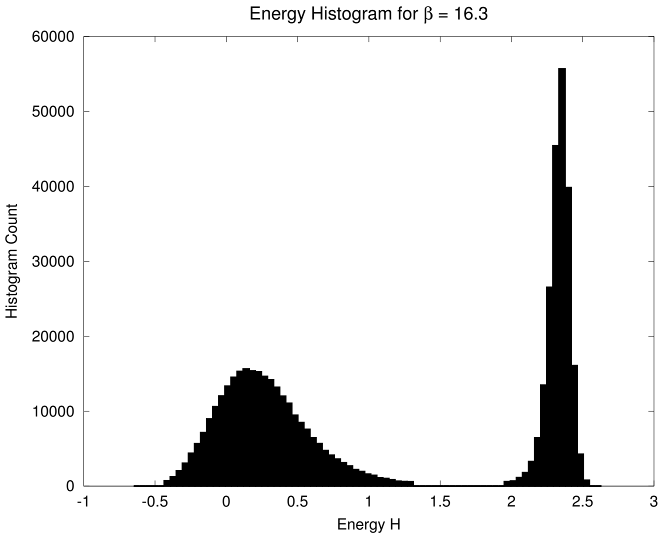
Custom Metropolis algorithm:

- Generate configurations sampled by π_T.
- Markov process, updating "current" configuration.
- We need $\frac{P(X')}{P(X)}$, but not $P(X)$ directly.
  $\rightarrow$ $\mathcal{Z}$ drops out!
- **This generates a "time series", but does not imply anything about real time evolution!**

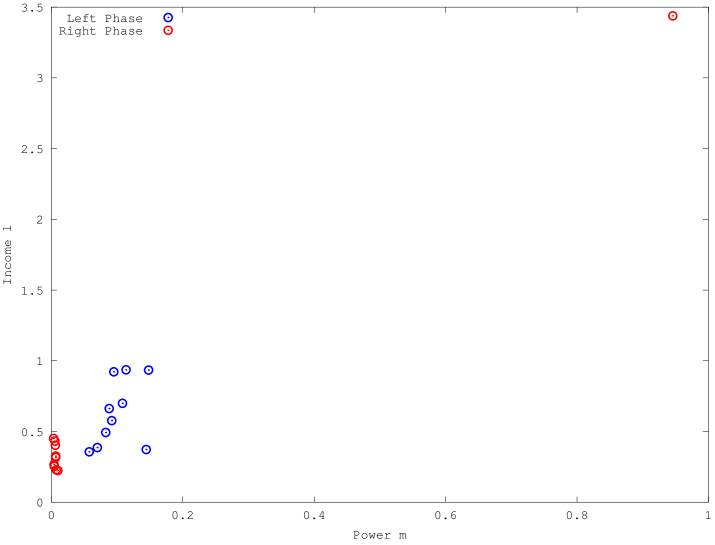
Energy Expectation



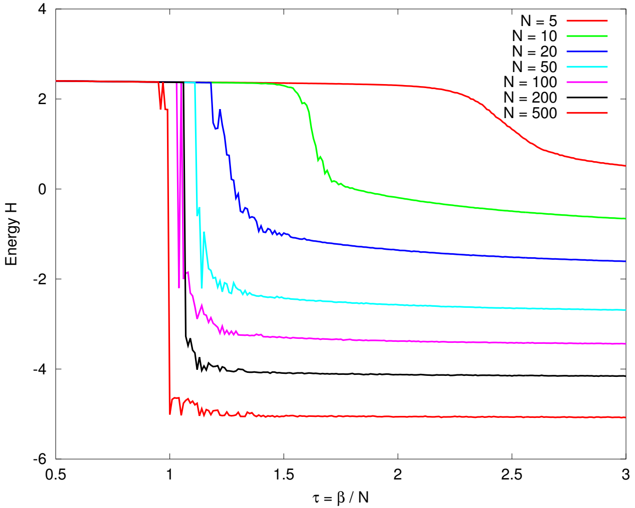
A Phase Transition



A Phase Transition



Infinite Volume Limit



Summary

- We set up a model describing individuals in a social space.
- It is *crucial* to model the **power distribution**!

Summary

- We set up a model describing individuals in a social space.
- It is *crucial* to model the **power distribution**!
- This model inherently shows **social inequality**.
- Transition happens as a **first-order phase transition**, breaking permutation symmetry spontaneously.

Summary

- We set up a model describing individuals in a social space.
- It is *crucial* to model the **power distribution**!
- This model inherently shows **social inequality**.
- Transition happens as a **first-order phase transition**, breaking permutation symmetry spontaneously.

Thanks for your attention!